


Exact Rank Reduction of Network Models

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With the advent of the big data era, generative models of complex networks are becoming elusive from direct computational simulation. We present an exact, linear-algebraic reduction scheme of generative models of networks. By exploiting the bilinear structure of the matrix representation of the generative model, we separate its null eigenspace and reduce the exact description of the generative model to a smaller vector space. After reduction, we group generative models in universality classes according to their rank and metric signature and work out, in a computationally affordable way, their relevant properties (e.g., spectrum). The reduction also provides the environment for a simplified computation of their properties. The proposed scheme works for any generative model admitting a matrix representation and will be very useful in the study of dynamical processes on networks, as well as in the understanding of generative models to come, according to the provided classification.

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I. INTRODUCTION

Network science is experiencing a burst of activity in the modeling and understanding of very large complex systems, including, for example, those formed by social interactions in microblogging platforms such as Twitter [1], or by high-throughput molecular biology data related to genomes [2], proteomes [3], metabolomes [4], etc. However, this endeavor is limited by the computational effort required to simulate dynamical processes running on top of very large real networks. Moreover, the full knowledge of the connectivity structure (links) and dynamic state of their units (nodes) is often unaffordable. In this case, the use of generative models of networks is the only alternative.

Generative models of networks are a powerful tool for studying real-world networks and the dynamical processes unfolding on them. They can provide insight into network formation by telling us which processes can (or cannot) lead to the development of certain descriptors. Their scope covers social networks [5], neuroscience [6], human mobility [7–9], finance [10], ecology [11,12], and more. Through networked models, researchers have improved the understanding of complex systems in terms of easily

interpretable analytical relations. A paramount example is the large literature on critical phenomena on complex networks: synchronization [13], spreading processes [14–20], or percolation [21–23], to cite a few. These models usually encode network structure into a small set of parameters and generative rules. In the past decades, along with the increased availability of highly resolved data, generative models of networks have flourished to explain newly observed properties like time-evolving contacts or multi-layer topologies. Generative models of networks have proven to be robust, accurate, and analytically treatable tools for describing families of networks, rather than single instances. We can say that they are still the tool of choice for uncovering mechanistic properties of complex systems that can be generalized to a wide set of contexts.

The main problem of this deluge of generative models is that, as they become richer and more intricate, their dimensionality increases, and they become harder to simulate and analyze. Furthermore, the lack of an inclusive theoretical framework makes it difficult to derive theoretical relationships among models, which could tell us about how instances of different generative models are similar in their structure and functionality.

We propose an exact rank-reduction scheme for the matrix representation of generative models of networks (network models henceforth). We reduce the effective dimensionality of network models, facilitating their static and dynamical analysis. Moreover, this scheme allows us to define universality classes of network models in terms of

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the reduced features. We build a general, algorithmic derivation of some of the most relevant properties of these models, as their spectrum, and the behavior of some linear and nonlinear dynamical systems coupled to them. We show that simple (low-rank) models explain the most important local properties present in real networks. We then define composition rules for models that allow us to reproduce more complex features, like mesoscale structures, while keeping the complexity of the models (in terms of number of variables necessary to reproduce the model) low. Finally, we describe the relationship between models, and define equivalence classes of models, in terms of the action of symmetry groups on the rank-reduced spaces. Our methodology provides a general framework for both classification and computation, that applies straightforwardly to future generative models, with no need to develop new *ad hoc* approaches.

II. CLASSIFICATION OF NETWORK MODELS

The configuration model is one of the simplest and most popular generative network models [24]. One fixes the expected degree of each node and considers all the network configurations (i.e., adjacency matrices) that respect the given degree sequence. This generalizes to more complex models, which are always made up of a set of properties, and an ensemble of network configurations, which is maximally entropic once the constraints induced by the properties hold. The entry A_{ij} of the matrix representation on such an ensemble is proportional to the probability that the link ij exists, and the properties defining the model completely determine the value of A_{ij} . In the configuration model of n nodes one has $A_{ij} = k_i k_j / (n \langle k \rangle)$, where k_i is the expected degree of node i and $\langle k \rangle$ is the expected average degree. We can write A in matrix form, $A = KK^T$, with K being the n -dimensional vector $K_i = k_i / \sqrt{n \langle k \rangle}$. Given that A is the outer product of K with itself, it is a rank-1 matrix: $\text{rank } A = 1$, no matter the size of the system (n).

We argue that the rank of the matrix A equals the number of node features the model constrains. The configuration model fixes only one feature per node—the expected degree—resulting in $\text{rank } A = 1$. A model fixing two node features would result in $\text{rank } A = 2$, and so on. As a result, any symmetric, $n \times n$, matrix with $\text{rank } A = r$ generates a model of a network of n nodes which constrains r properties per node. The general form of such a matrix is the linear combination of all the possible outer products among r linearly independent vectors. We call them metadegrees, as they generalize the degree vector of the configuration model:

$$A = \sum_{\mu, \nu=1}^r \Delta_{\mu\nu} v_\mu v_\nu^T. \quad (1)$$

where v_μ is an n -dimensional vector and represents the μ th metadegree, with $\mu = 1, \dots, r$. Δ is an $r \times r$ nonsingular

matrix and encodes the coefficients of mixing among metadegrees. The metadegrees can be arranged as columns of an $n \times r$ matrix V whose entry $V_{i\mu}$ represents the value of the μ th metadegrees for the i th node, leading to a pure matrix representation of A :

$$A = V\Delta V^T. \quad (2)$$

Equations (1) and (2) hold for any value of $r = 1, \dots, n$. However, we argue that r must be much smaller than the size of the system ($r \ll n$), as useful physical models are usually designed to depend on few—fundamental—parameters, compared to the size and complexity of the system under study. For this reason, we use this decomposition to classify and easily solve large network models using their low-rank linear algebraic structure. Notwithstanding, we remark that models featuring nonlocal and mesoscale structures (like clustering, modularity, bipartiteness) apparently violate our statement, as they need high rank, some even a rank growing with the size of the system n . We show that they can nonetheless be reduced to low-rank structures in Sec. V.

The decomposition in Eq. (2) entails a powerful interpretation of the matrix representation of a generative model as a bilinear form \mathbb{R}^n . Given two vectors $X, Y \in \mathbb{R}^n$, the matrix representation returns their scalar product $X^T A Y$. This connection between the matrix representation and the structure of a scalar product on \mathbb{R}^n , already unveiled in Ref. [25], has deep implications in our study.

The main feature of our scalar product, however, is being highly degenerate. The rank-nullity theorem tells us that the eigenspace of A associated to the eigenvalue 0 has dimension $n - r$. This eigenspace, which we call L , is completely determined in terms of the kernel of the linear map $\xi: \mathbb{R}^n \rightarrow \mathbb{R}^r$, defined in terms of V^T : $\xi(x) = V^T x$. In other words, L is the set of vectors $x \in \mathbb{R}^n$ for which $V^T x = 0$. L is the trivial subspace of \mathbb{R}^n . We focus on the restriction of ξ to the subspace on which it is invertible. There, it induces an isomorphism between a small subspace R of \mathbb{R}^n , with dimension r , and \mathbb{R}^r .

Packing all this together, we induce a decomposition of the full space, $\mathbb{R}^n \simeq R \oplus L$, and this is the core step in our rank reduction. Given that an isomorphism is also a change of basis, Δ is the representation in \mathbb{R}^r of the same scalar product that A is in R . Moreover, given that by definition the restriction of A to L is identically zero, Δ encodes exactly the same information as A , once the trivial degeneracy is pruned.

By Sylvester's law of inertia, we can always convert Δ to normal form, i.e., a diagonal matrix with entries 1s and -1 s (the number of positive entries of Δ is known as metric signature). Consequently, Eq. (2) is simply the scalar product A written into normal form. A schematic representation of the exact reduction is presented in Fig. 1.

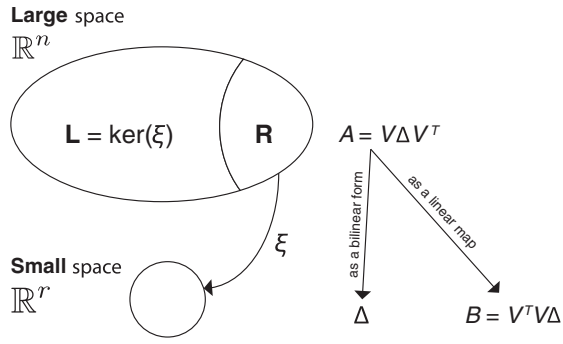


FIG. 1. Representation of the exact rank-reduction scheme. The transformation laws for the matrix representation of the generative model are also shown.

This decomposition has a straightforward consequence. It provides a universal classification of any possible matrix representation of a generative model for networks. All the possible network models (representable in terms of matrices) fall into classes defined by the rank r and the metric signature p of Δ . We denote these classes as (\mathbf{r}, \mathbf{p}) . Within each universality class, the values of metadegrees (columns of V) characterize a specific model, up to degeneracy induced by symmetry that we study in Sec. IV.

When $r = 1$, the only existing class is (1.1) and it contains the configuration model. At $r = 2$, we find two classes, which we can identify as Euclidean models (2.2) and Lorentzian models (2.1), according to the terminology used in general relativity.

A. Classification of popular generative models of networks

Class 2.1 is particularly interesting, as it includes the activity-driven model (ADM) [26], a widely used model for time-evolving networks. This model assigns each node a probability of activation a_i . When active, a node establishes links with m other random nodes (active or inactive). All links are reset before the next time step. The activity-driven model is an extremely simple model of temporal networks, yet, just as the configuration model, it has been successfully applied to many different contexts and exhibits a rich and interesting macroscopic behavior. Given the absence of temporal correlations, it is fully represented by the matrix $A = (m/n)(\Omega F^T + F \Omega^T)$, with $\Omega_i = a_i$ and $F_i = 1$; see Ref. [27]. From the previous expression, the rank-2 structure becomes apparent, as A is the outer product of two linearly independent vectors. To show its signature, we have to write the metadegrees with Δ in normal form [$\Delta = \text{diag}(1, -1)$]:

$$v_1 = \sqrt{\frac{m}{2n}} \langle a^2 \rangle^{1/4} \left(F + \frac{\Omega}{\sqrt{\langle a^2 \rangle}} \right), \quad (3)$$

$$v_2 = \sqrt{\frac{m}{2n}} \langle a^2 \rangle^{1/4} \left(F - \frac{\Omega}{\sqrt{\langle a^2 \rangle}} \right). \quad (4)$$

In a recent extension of the activity-driven model intended to mimic preferentiality in attachment [28], nodes are assigned specific values of attractiveness, in addition to the activity potential a_i . When a node activates, it will then be more likely to choose nodes with high attractiveness. This model falls again in class (2.1), and its reduced rank representation is the same as in Eq. (4), with a vector proportional to the attractiveness instead of the constant vector F of the original activity model (see the Appendix A). Also another extension, the simplicial activity-driven model [29], where active nodes create cliques, instead of single links, to account for multiagent interactions, can be accommodated in our classification. Depending on the relation between the average activity and the clique size, it can be easily proven that the simplicial activity-driven model is Euclidean (2.2) or Lorentzian (2.1). Specifically, one can prove that it is Euclidean if the average node activity $\langle a \rangle$ is higher than a threshold value: $\langle a \rangle > [2(q-2)]^{-1}$, where q is the size of the clique. When clique size is not fixed, but follows a given distribution, the threshold value is more involute but can still be computed. Appendix A contains details of the computation regarding the activity-driven model and its generalizations.

The configuration model with degree-degree correlations also falls in the rank $r = 2$ universality class. By setting the first metadegree vector of the model to be the degree vector K , one can make the Euclidean model 2.2 exhibit arbitrarily disassortative or assortative behavior [30] by tuning the second metadegree. Appendix B reports the details of the calculations.

Finally, we can rank reduce another popular generative model: the celebrated stochastic block model (SBM) [31]. It has a wide range of applications, as, for instance, community detection [32]. In its simpler form, nodes are divided into c subsets. Links within and between subsets occur with different probabilities. We define k to be the average number of connections a node establishes with nodes from the same subset, and h from subsets other than its own. One can show that the rank of the resulting matrix representation is equal to the number of subsets: $r = c$. The signature exhibits instead two regimes. If subsets reflect a community structure, and nodes are more likely to connect within the same subset [$k > h/(c-1)$], then the model is Euclidean, and belongs to the class (c.c). If, instead, nodes tend to form links across subsets [$k < h/(c-1)$], the metric has Lorentzian signature, and the model belongs to the family class (c.1). Note that this particular classification of stochastic block models separates modular structure from bipartite structures. Even in the presence of a fine partition (large c), the only possible signatures are

either Euclidean or Lorentzian. This will prove useful in what follows, when we study the internal symmetries of the different models. Appendix C contains the details of the calculation.

III. DERIVATION OF MODEL PROPERTIES

In addition to the classification, our rank-reduction scheme allows us to derive key properties of the models from their general form [Eq. (2)]. Therefore, the results apply straightforwardly to all models—already discovered or unknown—with no further need for *ad hoc* approaches. In what follows, we investigate the spectrum, as well as the solution of some linear and nonlinear processes. We start from the spectrum of the matrix representation of the model, which plays a key role in many centrality measures and determines the behavior of several dynamical processes. The largest eigenvalue [33], for instance, determines the critical behavior of synchronization [13] and diffusion [17,34–36] phenomena. We wonder if our rank reduction preserves the spectrum. We focus on the eigendecomposition of A in the subspace R , which is itself the direct sum of the eigenspaces of A relative to its nonzero eigenvectors. As mentioned, Eq. (2) is a change of basis for a bilinear form. Given that now we wish to preserve the spectrum, we need to treat A as a linear map $\mathbb{R}^n \rightarrow \mathbb{R}^n$. The new representation in \mathbb{R}^r is $B = \xi A \xi^{-1} = J \Delta$, with $J = V^T V$ (see also Fig. 1). The explicit expression of the inverse isomorphism is $\xi^{-1} = V J^{-1}$. Note that matrix J encodes all the possible scalar products among the metadegrees: $J_{\mu\nu} = v_\mu \cdot v_\nu$. We can give J a useful statistical interpretation. Let us assume the values of metadegrees of each node ($v_{\mu,i} = V_{i\mu}$) come from a given probability distribution: the metadegree distribution. The metadegree distribution is the generalization of the degree distribution beyond rank $r = 1$. Then, node metadegrees are samples from the metadegree distribution. As a result, the scalar product between v_μ and v_ν is proportional to the sample estimate of the expectation value of the product of these two metadegrees, intended as stochastic variables. In the limit of large network ($n \rightarrow \infty$), the sample estimate converges to the true expected value: $(v_\mu \cdot v_\nu)/n \rightarrow \langle v_\mu v_\nu \rangle$. This implies that the eigenvalues of the matrix representation of the generative model are linear combinations of the second moments of the metadegrees. This feature is extremely relevant, for instance, in the context of epidemic spreading. Many seminal works have shown the epidemic threshold of the configuration model to depend on the second moment of the degree [14,37], with important implications for disease containment. More recently, the same property was found for the activity distribution and other models [38]. We now discover that this is a general property of any model, not a peculiarity of those two.

Through rank reduction we can derive a simple formula for the eigenvectors, too. Let $\Lambda_\mu \neq 0$ be an eigenvalue of

A (and B) and let $f^{(\mu)}$ be an associate eigenvector of B in the reduced space ($B f^{(\mu)} = \Lambda_\mu f^{(\mu)}$). Then, $g^{(\mu)} = V \Delta f^{(\mu)}$ will be an eigenvector of A for the same eigenvalue. The eigenvalues and eigenvectors of B thus completely determine the spectral decomposition of A . Furthermore, one can choose $\{f^{(\mu)}\}$ to be a specific orthogonal basis of \mathbb{R}^r with respect to the scalar product Δ : $f^{(\mu)T} \Delta f^{(\nu)} = \delta_{\mu\nu} \Lambda_\mu^{-1}$. Then, $\{g^{(\mu)}\}$ will automatically be an orthonormal basis of R , thus completing an algorithmic construction of the spectral decomposition of the matrix from its rank-reduced transform.

In addition, the spectrum of A completely determines the behavior of any linear diffusion process of the form $\dot{x} = (a + bA)x$, which can then be solved in the reduced space and then projected back. We can, however, use rank reduction to solve a large class of nonlinear dynamical processes, too. Consider the following equation for the operator $X(t) \in \mathbb{R}^{n,n}$ (an $n \times n$ matrix):

$$\dot{X} = cX + dAX + Xf(AX), \quad (5)$$

where $c, d \in \mathbb{R}$, and f is an arbitrary holomorphic function, with $f(0) = 0$. This, for instance, includes spreadinglike quadratic terms [$\dot{X} = -cX + d(1 - X)AX$]. This equation contains all nonlinear terms in the form $X(AX)^j$, for any $j \in \mathbb{N}_0$. The operator $P = \xi^{-1}\xi$ is an orthogonal projector on the subspace R ; using P , we decompose X in terms of its action on the two subspaces L and R (remember Fig. 1), $X = X_{RR} + X_{RL} + X_{LR} + X_{LL}$: $X_{RR} = PXP$, $X_{RL} = PA(1 - P)$, and so on. By definition of R and L , A is nonzero only inside R : $A = PAP$. Using this, and the McLaurin decomposition of f [$f(z) = \sum_j f_j z^j$], we can prove that $f(AX) = Pf(AX_{RR}) + Pg(AX_{RR})AX_{RL}$. Note that g is another holomorphic function defined as $g(z) = \sum_j f_{j+1} z^j$. We can now decompose Eq. (5) in terms of the four parts of X :

$$\dot{X}_{RR} = cX_{RR} + dAX_{RR} + X_{RR}f(AX_{RR}), \quad (6)$$

$$\dot{X}_{RL} = \{c + [d + X_{RR}g(AX_{RR})]A\}X_{RL}, \quad (7)$$

$$\dot{X}_{LR} = X_{LR}[c + f(AX_{RR})], \quad (8)$$

$$\dot{X}_{LL} = cX_{LL} + X_{LR}g(AX_{RR})AX_{RL}. \quad (9)$$

Equation (6) is completely restricted to the reduced subspace R , and we can use our mapping ξ to send it to \mathbb{R}^r , by defining $U = \xi X \xi^{-1}$. The resulting equation is $\dot{U} = cU + dAU + Uf(BU)$, identical to Eq. (5), but living in the reduced space. Once we solve for U (either analytically or numerically, depending on the specific equation), we can go back to X_{RR} by using the inverse transformation. Once X_{RR} is known, Eqs. (7)–(9) are just linear, and can be solved with standard techniques

[39–42]. Remarkably, if we assume a simple (and often realistic) initial condition of $X(0) = \mathbb{I}$, then X_{LR}, X_{RL} are identically zero. This allows us to write a simple, explicit solution of Eq. (5): $X(t) = \xi^{-1}U(t)\xi + e^{ct}(1 - \xi^{-1}\xi)$. We have transformed a system of n^2 coupled nonlinear differential equations in n^2 unknowns into one of just r^2 . The gain is dramatic considering that n is the number of nodes (large), while r (the reduced rank) is for most known models very small. Appendix D contains the detailed solution of Eq. (5).

IV. SYMMETRIES IN THE SPACE OF GENERATIVE MODELS OF NETWORKS

Models in the same class \mathbf{r}, \mathbf{p} , with different metadegrees, may still be the same model in disguise. The action of representations of specific isometry groups in the small (\mathbb{R}^r) and large spaces (\mathbb{R}^n) may induce isomorphisms between models previously considered as separate physical objects. First, we consider the isometry group of Δ : $\text{Iso}(\Delta)$. The $r \times r$ matrix Q belongs to the representation of $\text{Iso}(\Delta)$ in \mathbb{R}^r if it leaves Δ unchanged: $Q^T \Delta Q = \Delta$. We call them *internal transformations* because of their action on the metadegrees: $v_{\mu,i} \rightarrow v'_{\mu,i} = \sum_{\nu} Q_{\mu\nu} v_{\nu,i}$. They happen inside a node: they mix its metadegree values, but do not mix metadegree values belonging to different nodes. Internal transformations leave A unchanged: $A = V \Delta V^T \rightarrow V Q^T \Delta Q V^T = A$. Hence, two models whose metadegrees are connected by an internal transformation are, for all intents and purposes, the same model. Internal symmetries do modify B , though clearly not its spectrum: $B \rightarrow Q B Q^{-1}$. We can then choose Q wisely, so that B has the simplest possible form, provided we know the structure of $\text{Iso}(\Delta)$. Luckily, we have shown that the known models have either a Euclidean or a Lorentzian signature, whose isometry groups are the most known and studied [43]: respectively, they are the orthogonal group and the r -dimensional Lorentz group. In the Euclidean case ($B = J$) we can go further, as there always exists an orthogonal matrix Q so that $Q B Q^{-1} = Q B Q^T = Q J Q^T$ is diagonal. This means that the rotated metadegrees $V Q^T$ are orthogonal, and since $\Delta = \mathbf{I}$, their norm directly gives the spectrum of A . As a result, whenever the metadegrees are orthogonal (or we can make them so with a rotation in \mathbb{R}^r), (i) their norms are the nonzero eigenvalues of A and (ii) the metadegrees are also eigenvectors of A . These isometries are of special interest for the understanding of geometrical embeddings of complex networks [44].

We now turn to external transformations. They act on the large space \mathbb{R}^n by mixing the values each metadegree has on the nodes. Opposite to internal transformations, they do mix nodes but do not mix different metadegrees. These transformations comprise the orthogonal group $O(n)$, and act on the metadegrees as follows: $V \rightarrow S V$, for $S \in O(n)$.

It is easy to show that they change A through a similarity transformation, while keeping B unchanged.

In summary, we have found that internal symmetries span different low-rank representations (B) of the same model (A). External transformations instead span all the models (A) that have the same low-rank representation (B). Both symmetries, however, preserve the spectrum. Internal and external symmetries are schematically represented in Fig. 2. We now study the combined action of internal and external symmetry transformations: $V \rightarrow S V Q^T$. In the Euclidean case, this coincides with the singular value decomposition of V . In general, it still has far-reaching implications on the nature of models themselves. Within the same class, it allows mapping of different models onto each other. Models that are completely different in nature and purpose may have the same properties if they are linked by this symmetry transformation. As a practical example, we now show that within class 2.1, the activity-driven model can be mapped onto the stochastic block model with two subsets and high intersubset connectivity ($h > k$) (SBM). This is quite remarkable if we consider that the former is commonly used to model time-evolving networks with fixed microscopic activity patterns, while the latter applies to static networks featuring mesoscale structures. We start from an ADM with fixed activity vector Ω and number of stubs m [see Eq. (4)]. We will land on a SBM featuring two equally sized subsets, whose degrees k, h will be computed. This means finding the transformation which gives $V_{\text{SBM}} = S V_{\text{ADM}} Q^T$, where $V_{\text{SBM}}, V_{\text{ADM}}$ are the metadegrees of the two models when the metric is in normal form [$\Delta = \text{diag}(1, -1)$]. The two-dimensional Lorentz group has one parameter and consists in hyperbolic rotations on the plane; its infinitesimal generator is the first Pauli matrix (σ_1). Hence, we can span the internal transformations using the hyperbolic angle θ and by exponentiation of the infinitesimal generator: $Q = e^{\theta \sigma_1}$. The transformation relation, made explicit for each of the two metadegrees, defining $h^+ = \sqrt{h+k}$ and $h^- = \sqrt{h-k}$, becomes

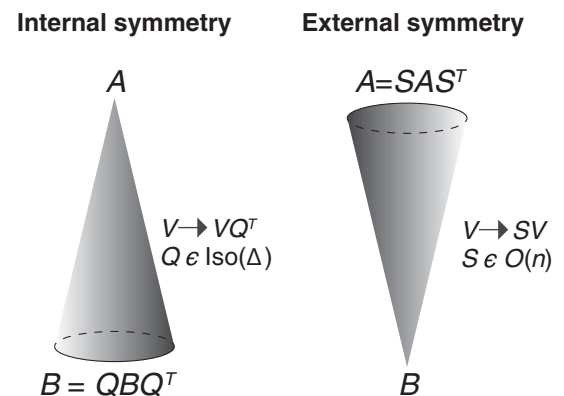


FIG. 2. Schematic representation of internal and external symmetries of the rank-reduction scheme.

$$\begin{aligned}(h^+ + h^-)F_1 + (h^+ - h^-)F_2 &= e^{-\theta}\sqrt{2m}SF, \\ (h^+ - h^-)F_1 + (h^+ + h^-)F_2 &= e^{\theta}\sqrt{2m}S\Omega,\end{aligned}\quad (10)$$

where the entries of vectors F_1, F_2 are 1 on the first (second) subset, zero otherwise, so that $F_1 + F_2 = F$. An explicit form of S would then solve Eq. (10). We, however, only wish to uncover under which conditions such mapping is possible. Thus, we just require that S , being orthogonal, preserve standard scalar products in \mathbb{R}^n . That fixes the degrees of the SBM: $k = m\langle a \rangle$, $h = m\sqrt{\langle a^2 \rangle}$. It also fixes the hyperbolic angle of the internal transformation to $\theta = -\frac{1}{4}\log\langle a^2 \rangle$. This demonstrates that we can indeed map the ADM onto a SBM, and that the mapping fixes the parameters of such SBM, and also fixes the gauge induced by internal transformations.

As a last comment, we point out that our classification scheme could be used to generate new models. Traditionally, models have been designed *ad hoc* to investigate specific network properties. Now, for each class \mathbf{r}, \mathbf{p} , one could enumerate all the potential models it contains, by means of a systematic characterization of its internal and external symmetries. Some of these models could feature previously unrepresented and unstudied network properties.

V. PRODUCT OF GENERATIVE MODELS OF NETWORKS

The stochastic block model, which we have already examined, features a rank that equals the number of subsets (or communities). If one needs a stochastic block model with many subsets (fine partition), the rank may then be large. This seems to go against our claim that our rank reduction is powerful because the typical required rank is small, and happens because of the presence of multiscale structures. We posit that if a model features organizational scales above node-node correlations (clustering, communities, multipartiteness, and so on), its rank needs to be large. It might even need to scale with the size of the system. We introduce here a new operation that allows us to overcome this apparent drawback by reducing high-rank structures to low-rank components. This will allow us to extend our classification to models that exhibit complex mesoscale features and to apply the full power of our machinery to them.

We define the tensor product of two models as the tensor (Kronecker) product of their matrix representation. We start from a model $A_1 = V_1\Delta_1V_1^T$, with n_1 nodes and rank and signature r_1, p_1 , and a model $A_2 = V_2\Delta_2V_2^T$, with n_2, r_2, p_2 , respectively. The product model $A = V\Delta V^T$ is

$$\begin{aligned}A &= A_1 \otimes A_2 = (V_1\Delta_1V_1^T) \otimes (V_2\Delta_2V_2^T) \\ &= (V_1 \otimes V_2)(\Delta_1 \otimes \Delta_2)(V_1 \otimes V_2)^T.\end{aligned}\quad (11)$$

Equation (11) shows that the metadegree matrix and the metric are simply the tensor product of the original matrices: $V = V_1 \otimes V_1^T$ and $\Delta = \Delta_1 \otimes \Delta_2$. The resulting network has $n = n_1n_2$ nodes, rank $r = r_1r_2$, and signature

$$p = r \left(1 - \frac{p_1}{r_1} - \frac{p_2}{r_2} + 2 \frac{p_1 p_2}{r_1 r_2} \right).\quad (12)$$

Appendix E contains the proof of Eq. (12).

A model which is a tensor product of two smaller models is thus completely defined and characterized by its factors. This is a further simplification as high-rank models can actually be studied in terms of their low-rank factors. The properties of the spectrum, linear and nonlinear dynamics, completely follow from the study of the factors. For instance, the spectrum of A is composed of all the possible products of one eigenvalue of A_1 with one eigenvalue of A_2 .

In general, the product decomposition of models simplifies the classification and treatment of high-rank models. Any model whose rank r is not a prime number can be decomposed into the product of τ lower-rank models, with τ the number of factors of r :

$$\mathbf{r}, \mathbf{p} = (\mathbf{r}_1, \mathbf{p}_1) \otimes (\mathbf{r}_2, \mathbf{p}_2) \otimes \cdots \otimes (\mathbf{r}_\tau, \mathbf{p}_\tau),\quad (13)$$

provided

$$r = \prod_{j=1}^{\tau} r_j,\quad (14)$$

$$p = \frac{1}{2} \left[r + \prod_{j=1}^{\tau} (2p_j - r_j) \right].\quad (15)$$

Appendix E contains the proof of Eq. (15).

A. Model entanglement

While a decomposition of the form in Eq. (13) always exists, Eqs. (14) and (15) are not sufficient for Eq. (13) to hold. In other words, a model may or may not be decomposed by a specific tensor product respecting Eqs. (14) and (15).

As an example, let us consider a generic model in 4.2. We can decompose it in two ways: as the product of a degree-degree correlated configuration model and an activity-driven model, $\mathbf{2}, \mathbf{2} \otimes \mathbf{2}, \mathbf{1}$, or as the product of two activity-driven models, $\mathbf{2}, \mathbf{1} \otimes \mathbf{2}, \mathbf{1}$. Let us choose the former. The general form of a model in 4.2 is $A = v_1v_1^T + v_2v_2^T - v_3v_3^T - v_4v_4^T$. As we proved before, the four metadegrees span a four-dimensional space isomorphic to \mathbb{R}^4 . We change the basis of such space to highlight a $\mathbb{R}^2 \otimes \mathbb{R}^2$ structure: $v_1 = e_1 \otimes f_1$, $v_2 = e_1 \otimes f_2$, $v_3 = e_2 \otimes f_1$, $v_4 = e_2 \otimes f_2$. By substituting it into the expression of the model, we get

$$\begin{aligned}
A &= \sum_{a,b,c,d=1}^2 \eta_{ab} \delta_{cd} (e_a \otimes f_c)(e_b \otimes f_d)^T \\
&= \sum_{a,b,c,d=1}^2 (\eta_{ab} e_a e_b^T) \otimes (\delta_{cd} f_c f_d^T), \quad (16)
\end{aligned}$$

where $\eta = \text{diag}(1, -1)$ is the two-dimensional Lorentz metric. Equation (16) highlights the decomposition into a model **2.2** ($\delta_{cd} f_c f_d^T$) and a model **2.1** ($\eta_{ab} e_a e_b^T$). However, if we choose a change of basis that mixes the metadegrees in some other way, the product might not factorize. This is equivalent to the entangled states in quantum mechanics. A model is separable with respect to two potential factors if it can be expressed in terms of a product of the two factors. It is entangled if this is not true. Inspired by the quantum mechanical analogy, we state that a model A is separable with respect to the product $A_1 \otimes A_2$ if and only if

$$A = \frac{\text{tr}_2 A \otimes \text{tr}_1 A}{\text{tr} A}, \quad (17)$$

where tr_1 (tr_2) is the trace of A in the space where A_1 (A_2) lives.

B. Creation of new multiscale structures

We have already shown that the stochastic block model with c subsets, n_c nodes per subset ($n_c = n/c$), belongs to either class **c.c** or **c.1**. The tensor decomposition provides a natural way to simplify it further, and then generalize it:

$$\text{SBM}(c = c, n_c = n_c) \equiv \text{SBM}(c = c, n_c = 1) \otimes \text{CM}, \quad (18)$$

where CM is the configuration model. The stochastic block model is then the product of the same model but with only one node per subset and a configuration model. The former describes between-subset connectivity, and the latter within-subset connectivity. From this relation, a very important interpretation of the product becomes apparent: The tensor product of models is deeply connected with a multiscale structure of the network. Once we formulate this through Eq. (18), we immediately understand that we do not need to stick to the configuration model for the within-subset connectivity: we can plug in whatever model we want. For instance, we can define a stochastic block model with an activity-driven model within the subsets simply as

$$\text{SBM}(c = c, n_c = n_c) = \text{SBM}(c = c, n_c = 1) \otimes \text{ADM}. \quad (19)$$

This model will have rank $r = 2c$ and signature $p = c$ regardless of the signature of the stochastic block model.

C. Kronecker graphs

The product of the models defined above bears a clear resemblance to Kronecker graphs [45–49]. Including Kronecker graphs into our classification scheme is important because of the nontrivial properties they exhibit, like clustering and k -core organization [50]. As we have seen, mesoscale structures in the network likely require a high rank, possibly scaling as the size of the system. This would seem as an apparent drawback of our theory. Including Kronecker graphs using model products solves that. Even if the rank of the full model scales with the number of nodes, the rank of the base model remains small, and this is the only thing we need to solve it. We set as the base of the Kronecker graph a small-rank model \hat{A} with \hat{n} , \hat{r} , \hat{p} . For a definition of the base (or initiator) of a Kronecker graph, see Refs. [45,51]. The Kronecker model A is then the tensor power,

$$A = \hat{A}^{[\tau]} = \underbrace{\hat{A} \otimes \hat{A} \otimes \cdots \otimes \hat{A}}_{\tau \text{ times}}, \quad (20)$$

with τ being the power of the model: $\tau = \log n / \log \hat{n}$. The metadegrees and metric of the full model are $V = \hat{V}^{[\tau]}$, $\Delta = \hat{\Delta}^{[\tau]}$. The rank and signature of the full model are

$$r = \hat{r}^\tau = n^{\log \hat{r} / \log \hat{n}}, \quad (21)$$

$$p = \frac{\hat{r}^\tau + (2\hat{p} - \hat{r})^\tau}{2}. \quad (22)$$

Equation (21) shows that the rank increases with the size of the system. Notwithstanding, the model is easy to solve because only to solve its base \hat{A} is needed. Appendix E contains the proof of Eq. (22).

VI. AN APPLICATION: COMMUNICABILITY

Finally, to highlight the practical advance we can directly draw from our proposal, let us comment on a simple application of this reduction to compute a magnitude that directly depends on the matrix form of a network, the communicability. Communicability C_{ij} accounts for all the possible walks, which are generalized paths allowing for repetition of nodes and edges—of any length (weighted in decreasing order of length)—that join i , j , and measures how easily information flows between the two nodes [52,53]. In its most common form, walks have a penalization equal to the inverse of the factorial of their length, so that communicability has the following closed form: $C = e^A$. As a practical application of our theory, we now study the communicability of models. In particular, we examine the relationship between local connectivity (A) and nonlocal connectivity (C).

Inserting Eq. (2) into the definition of the communicability, we get

$$C = \sum_{m=0}^{\infty} \frac{1}{m!} (V\Delta V^T)^m = 1 + V\Delta \sum_{m=1}^{\infty} \frac{1}{m!} B^{m-1} V^T, \\ C = 1 + V\Delta B^{-1}(e^B - 1)V^T = 1 + V\Psi V^T, \quad (23)$$

where $\Psi = \Delta B^{-1}(e^B - 1)$ and $\Psi \in \mathbb{R}^{r,r}$. C has maximal rank (n), as it is the exponential of A . However, rank reduction returns a very simple form. The high-rank structure is simply represented by the identity matrix. Intuitively, this is the exponentiation of the large trivial subspace with eigenvector zero. The nontrivial part of the exponentiation (Ψ) occurs within the small space, as Eq. (23) shows. This means that communicability can be rank reduced, and computed in the small space. Notably, Ψ is also a well-behaved metric in \mathbb{R}^r (it is symmetric), and $V\Psi V^T$ is itself a network model [compare it with Eq. (2)], with rank equal to r , as Ψ is nonsingular. This implies that the communicability C of a model A in $\mathbf{r.p}$ is itself a model with the same rank r . What about the signature of this new model? In principle, communicability may fall into any class $\mathbf{r.p}'$. If B and Δ commute, then it is easy to prove that $p' = p$. In any other case, communicability may change universality class with respect to the original model. Commutation between B , Δ is guaranteed when the original model is Euclidean ($p = r$), or, in general, whenever the metadegrees corresponding to different signs in Δ are orthogonal. In other words, $v_\mu \cdot v_\nu \neq 0$ only if $\Delta_{\mu\mu} = \Delta_{\nu\nu}$ (with Δ in normal form).

In summary, the practical implications are the following. (i) One can express the nonlocal connectivity patterns (communicability) of any network model, in terms of the local connectivity (adjacency matrix) of a model with the same rank as the original one. (ii) In general, the universality class of the communicability and the one of the original model are different ($p' \neq p$). This means that nonlocal connectivity patterns exhibit a topology that is qualitatively different from local ones: communicability is a nontrivial emergent property of the local model. (iii) Rank reduction allows one to build network ensembles with given communicability. We do not explore this in the present paper, but we think it can lead to promising practical implications. (iv) In some specific cases (Euclidean models, models that do not mix different signature subspaces), the communicability is in the same universality class of the original model, hinting at the fact that these models might exhibit some forms of scale invariance.

To conclude this application, we explicitly compute the communicability of the configuration model and the activity-driven model:

$$C_{ij}^{(\text{CM})} = \delta_{ij} + \frac{\langle k \rangle}{\langle k^2 \rangle} (e^{\langle k^2 \rangle / \langle k \rangle} - 1) \frac{k_i k_j}{n \langle k \rangle}; \quad (24)$$

the communicability of the configuration model is proportional to its adjacency matrix. We now turn to the

activity-driven model (2.1), with metadegrees as in Eq. (4), and $\Delta = \text{diag}(1, -1)$. Given that $v_1 \cdot v_2 = 0$, the communicability of this model is also 2.1. We first compute Ψ :

$$\Psi_{\text{ADM}} = \frac{1}{m} \begin{pmatrix} \frac{e^{m(\sqrt{\langle a^2 \rangle} + \langle a \rangle)} - 1}{\sqrt{\langle a^2 \rangle} + \langle a \rangle} & 0 \\ 0 & \frac{e^{-m(\sqrt{\langle a^2 \rangle} - \langle a \rangle)} - 1}{\sqrt{\langle a^2 \rangle} - \langle a \rangle} \end{pmatrix}. \quad (25)$$

For brevity, we define

$$\psi_1 = \frac{e^{m(\sqrt{\langle a^2 \rangle} + \langle a \rangle)} - 1}{\sqrt{\langle a^2 \rangle} + \langle a \rangle}, \quad (26)$$

$$\psi_2 = \frac{e^{-m(\sqrt{\langle a^2 \rangle} - \langle a \rangle)} - 1}{\sqrt{\langle a^2 \rangle} - \langle a \rangle}. \quad (27)$$

We can now write the communicability matrix for the activity-driven model:

$$C_{ij}^{(\text{ADM})} = \delta_{ij} + \frac{1}{2n} \left[(\psi_1 + \psi_2) \left(\sqrt{\langle a^2 \rangle} + \frac{a_i a_j}{\sqrt{\langle a^2 \rangle}} \right) + (\psi_1 - \psi_2)(a_i + a_j) \right]. \quad (28)$$

Notably, the communicability of the basic activity-driven model exhibits the diagonal terms FF^T and $\Omega\Omega^T$ like the adjacency matrix of the simplicial activity-driven model (see Appendix A 1).

VII. CONCLUSION

We propose a linear algebraic methodology for classifying generative network models. By using the rank reduction of the matrix representation of generative models of networks, we derive many of their properties, and the properties of the dynamical processes on top of them. We do this by solving the generic model, so that our results apply algorithmically to all specific cases, with a dramatic decrease in the complexity of the analytic and numerical calculations involved. Our scheme includes models featuring many properties that are needed to study real datasets. These are both local properties, like node degree or activity, as well as more complex nonlocal features, like mesoscale structures. Finally, we show how the geometrical properties of our scheme can be used to devise new models and to extend and make connections between existing ones.

Some network structures still elude our rank-reduction scheme. We do not account for regular topologies (like lattices). Neither do we cover finite-size effects (n far from the thermodynamic limit). Time-evolving network models featuring temporal correlations are also not included at the moment, but we are confident that future formulations will

extend to them. For example, temporal correlations could emerge out of tensor representations of models [25,54], and finite-size effects as perturbations of the spectrum of A with diagonal matrices [55]. These are just ideas we are considering for a future work.

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APPENDIX A: ACTIVITY-DRIVEN MODEL

We start from the generalized activity-driven model presented in Ref. [28], of which the original formulation of Ref. [26] is a special case.

The model fixes two features for each node: the activity rate a_i and the attractiveness b_i . A node activates at a rate a_i and, when active, it links to m other nodes chosen with probability proportional to their attractiveness b_j . The expected value of each link thus is

$$A_{ij} = \frac{m}{n\langle b \rangle} (a_i b_j + a_j b_i). \quad (\text{A1})$$

We arrange activity rates in vector Ω : $\Omega_i = a_i$. We arrange attractiveness values in vector Σ : $\Sigma_i = b_i/\langle b \rangle$. This results in

$$A = \frac{m}{n} (\Omega \Sigma^T + \Sigma \Omega^T), \quad (\text{A2})$$

proving that the activity-driven model has rank $r = 2$. We now diagonalize the metrics and get to the metadegrees. We define

$$U_1 = \left(\frac{m}{2n\langle b \rangle} \right)^{1/2} \left(\frac{\langle b^2 \rangle}{\langle a^2 \rangle} \right)^{1/4} \left[\Omega + \left(\frac{\langle b^2 \rangle}{\langle a^2 \rangle} \right)^{-1/2} \Sigma \right], \quad (\text{A3})$$

$$U_2 = \left(\frac{m}{2n\langle b \rangle} \right)^{1/2} \left(\frac{\langle b^2 \rangle}{\langle a^2 \rangle} \right)^{1/4} \left[\Omega - \left(\frac{\langle b^2 \rangle}{\langle a^2 \rangle} \right)^{-1/2} \Sigma \right]. \quad (\text{A4})$$

These vectors diagonalize the expression in Eq. (A2):

$$A = \frac{m}{n} (U_1 U_1^T - U_2 U_2^T). \quad (\text{A5})$$

U_1, U_2 are metadegrees of the model and they show the signature $p = 1$. In addition, their specific normalization diagonalizes B , as $U_1 \dot{U}_2 = 0$. In the formalism of Sec. IV we have used the degeneracy induced by internal

transformation to choose metadegrees which automatically diagonalize B . Because of this, the two nonzero eigenvalues of A are $(m/n)\|U_1\|^2$ and $-(m/n)\|U_2\|^2$; i.e.,

$$\frac{m}{\langle b \rangle} \left(\langle ab \rangle \pm \sqrt{\langle a^2 \rangle \langle b^2 \rangle} \right). \quad (\text{A6})$$

Setting $b_i = 1$ recovers the original version of the activity-driven model:

$$A_{ij} = \frac{m}{n} (a_i + a_j). \quad (\text{A7})$$

$$m \left(\langle a \rangle \pm \sqrt{\langle a^2 \rangle} \right). \quad (\text{A8})$$

Compare the above result with Eq. (4) of Ref. [26].

1. Simplicial activity-driven model

In the version of the activity-driven model introduced in Ref. [29], an active node builds a clique (complete subgraph) with other $q - 1$ random nodes. It thus generates a q clique. q is sampled from a given distribution. The value $q = 2$ recovers the standard activity-driven model (with $m = 1$).

One can show that the simplicial structure induces a correction to Eq. (A7):

$$A_{ij} = \frac{\langle q \rangle - 1}{n} (a_i + a_j) + \frac{\langle a(q-1)(q-2) \rangle}{n} (1 - a_i)(1 - a_j). \quad (\text{A9})$$

The matrix form of A is

$$A = \frac{\langle q \rangle - \langle a(q-1)(q-2) \rangle - 1}{n} (\Omega F^T + F \Omega^T) + \frac{\langle a(q-1)(q-2) \rangle}{n} (F F^T + \Omega \Omega^T), \quad (\text{A10})$$

where we remind the reader that F is the n -dimensional vector of ones: $F_i = 1$. From this expression the (non-normal) metric is

$$\Delta = \begin{pmatrix} \frac{\langle a(q-1)(q-2) \rangle}{n} & \frac{\langle q \rangle - \langle a(q-1)(q-2) \rangle - 1}{n} \\ \frac{\langle q \rangle - \langle a(q-1)(q-2) \rangle - 1}{n} & \frac{\langle a(q-1)(q-2) \rangle}{n} \end{pmatrix}. \quad (\text{A11})$$

The eigenvalues of this matrix are

$$\text{eigenvalue}_1 = \frac{\langle q \rangle - 1}{n}, \quad (\text{A12})$$

$$\text{eigenvalue}_2 = \frac{1 - \langle q \rangle + 2\langle a(q-1)(q-2) \rangle}{n}. \quad (\text{A13})$$

The first eigenvalue is always positive, given that $q \geq 2$. The second eigenvalue is positive if and only if $\langle a(q-1)(q-2) \rangle > (\langle q \rangle - 1)/2$. When this relation among $\langle aq^2 \rangle$, $\langle aq \rangle$, $\langle a \rangle$, $\langle q \rangle$ holds, the model is **2.2**. Otherwise, it is **2.1**. When instead $\langle a(q-1)(q-2) \rangle = (\langle q \rangle - 1)/2$, the model is a rank-1 configuration model. This proves that the model is characterizable and solvable for a generic joint distribution of q , a . For simplicity, we now carry out the calculation in the simple case of constant q . The form of A is

$$A = \frac{q-1}{n} \left(1 - \frac{\langle a \rangle}{2a_c} \right) (\Omega F^T + F \Omega^T) + \frac{q-1}{n} \frac{\langle a \rangle}{2a_c} (F F^T + \Omega \Omega^T), \quad (\text{A14})$$

where we explicitly highlighted a critical activity value: $a_c = [2(q-2)]^{-1}$. Equation (A14) clearly shows that a_c discriminates between a regime where the Lorentzian part dominates and another where the Euclidean part dominates. Indeed, for high average activity, $\langle a \rangle > a_c$, the model is Euclidean (**2.2**). For low average activity $\langle a \rangle < a_c$, the model is Lorentzian (**2.1**), like the standard activity-driven model. Instead, when $\langle a \rangle = a_c$, the model collapses onto a rank $r = 1$ configuration model with node degree proportional to $1 + a_i$.

Now that the classification is complete, one can compute the eigenvalues of A by computing B , getting the same threshold condition as in Ref. [29].

APPENDIX B: CONFIGURATION MODEL WITH DEGREE-DEGREE CORRELATIONS

The rank $r = 2$ Euclidean class **2.2** is the simplest (min rank, Euclidean) model featuring degree-degree correlations. It exhibits arbitrary assortative or disassortative behavior. The general (diagonal) form of this model is

$$A = \frac{1}{n} (u u^T + v v^T), \quad (\text{B1})$$

where u , v are its two metadegrees. We have made the dependence of metadegrees on system size explicit. We change the metadegree basis so that one of them is the degree vector $k = \langle u \rangle u + \langle v \rangle v$ (where k_i is the degree of node i). To do this we set $v = (k - \langle u \rangle u) / \sqrt{\langle k \rangle - \langle u \rangle^2}$, provided the expression inside the square root is positive. The matrix representation is now

$$A = \frac{1}{n} \frac{1}{\langle k \rangle - \langle u \rangle^2} [\langle k \rangle u u^T + k k^T - \langle u \rangle (u k^T + k u^T)]. \quad (\text{B2})$$

Following Ref. [56], we measure the assortativeness as

$$\mathcal{A}(k) = \left\langle \frac{dk^{(nn)}}{dk} \right\rangle, \quad (\text{B3})$$

where $k_i^{(nn)} = \sum_{j \neq i} A_{ij} k_j / k_i$ is the expected degree of the neighbors of i . Using Eq. (B2), we compute

$$k^{(nn)} = ak + bu, \quad (\text{B4})$$

with

$$a = \frac{\langle k^2 \rangle - \langle u \rangle \langle uk \rangle}{\langle k \rangle - \langle u \rangle^2}, \quad (\text{B5})$$

$$b = \frac{-\langle u \rangle \langle k^2 \rangle + \langle k \rangle \langle uk \rangle}{\langle k \rangle - \langle u \rangle^2}. \quad (\text{B6})$$

This results in

$$\mathcal{A}(k) = \frac{b}{k} \left(\frac{du}{dk} - \frac{u}{k} \right). \quad (\text{B7})$$

Assuming a constant assortativity $\mathcal{A}(k) = \mathcal{A}$, one can solve the differential equation in $u(k)$, getting

$$u(k) = \left(u_0 + \frac{\mathcal{A}}{b} \log k \right) k, \quad (\text{B8})$$

where $u_0 = u(1)$. From this we see that the model **2.2** allows any arbitrary value of assortative ($\mathcal{A} > 0$) or disassortative ($\mathcal{A} < 0$) behavior, by tuning the dependence of the second metadegree u on the degree. Furthermore, Eq. (B8) tells us that for no assortativity ($\mathcal{A} = 0$), the model reduces to **1.1**—the rank $r = 1$ configuration model—because $u = u_0 k$ means that u , k are no longer linearly independent. Instead, we see that degree-degree correlations arise from a logarithmic correction to the $\mathcal{A} = 0$ solution. The free parameter u_0 is fixed by consistency, as b contains moments of u , k whose value must be compatible with the solution.

APPENDIX C: STOCHASTIC BLOCK MODEL

The matrix representation of the model is

$$A = \frac{ck}{n} \sum_{\mu=1}^c F_{\mu} F_{\mu}^T + \frac{ch}{n(c-1)} \sum_{\mu \neq \nu} F_{\mu} F_{\nu}^T, \quad (\text{C1})$$

$$A = \frac{c}{n} \left\{ \left[k - \frac{h}{(c-1)} \right] \sum_{\mu=1}^c F_{\mu} F_{\mu}^T + \frac{h}{(c-1)} \sum_{\mu, \nu=1}^c F_{\mu} F_{\nu}^T \right\}, \quad (\text{C2})$$

$$A = \frac{c}{n} \left\{ \left[k - \frac{h}{(c-1)} \right] \sum_{\mu=1}^c F_{\mu} F_{\mu}^T + \frac{h}{(c-1)} F F^T \right\}, \quad (\text{C3})$$

where F_{μ} has entries equal to one corresponding to subset $\mu = 1, \dots, c$, zero otherwise. These vectors $\{(c/n)F_1, (c/n)F_2, \dots, (c/n)F_c\}$ are linearly independent. This proves the rank $r = c$. We choose them as metadegrees. The metric (not in normal form) is

$$\Delta_{\mu\mu} = k, \quad (\text{C4})$$

$$\Delta_{\mu\nu} = \frac{h}{c-1}, \quad \text{for } \mu \neq \nu. \quad (\text{C5})$$

This matrix has one eigenvalue $k + h$ with multiplicity $c - 1$, which is always positive. It also has one eigenvalue $k - h/(c - 1)$, with multiplicity one. Hence, if $k > h/(c - 1)$, the signature is $p = r = c$ (Euclidean). If $k < h/(c - 1)$, the signature is $p = 1$ (Lorentzian).

APPENDIX D: NONLINEAR DYNAMICS

We detail the solution of Eq. (5), in terms of the solution of the system in Eqs. (6)–(9). Equation (6) is the only one containing the nonlinearity, but it is fully contained in the small space \mathbb{R}^f . We solve that for X_{RR} (either analytically or numerically), as explained in the main text [$X_{RR}(t) = \xi^{-1}U(t)\xi$]; the other equations then become linear. Therefore, they are always solvable [40,42], using series expansions like Dyson's [39].

Before proceeding any further, we recall the definition of Dyson's time-ordering operator \mathcal{T} :

$$\begin{aligned} \mathcal{T}[X(t_1)X(t_2)] &= \theta(t_1 - t_2)X(t_1)X(t_2) \\ &\quad + \theta(t_2 - t_1)X(t_2)X(t_1), \end{aligned} \quad (\text{D1})$$

with θ being Heaviside's step function. We also define Dyson's time-ordered exponentiation:

$$\begin{aligned} &\mathcal{T} \exp\left(\int_0^t ds X(s)\right) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \int_0^t ds_1 ds_2 \dots ds_m \mathcal{T}[X(s_1)X(s_2)\dots X(s_m)] \quad (\text{D2}) \\ &= \sum_{m=0}^{\infty} \int_0^t ds_1 \int_0^{s_1} ds_2 \dots \int_0^{s_{m-1}} ds_m X(s_m)\dots X(s_2)X(s_1). \end{aligned} \quad (\text{D3})$$

We now start from Eq. (7). It is a nonautonomous linear system with unknown X_{RL} . We recall that X_{RR} is no longer

unknown, as it is fully determined by Eq. (6). Its solution in terms of Dyson's series is

$$\begin{aligned} X_{RL}(t) &= \mathcal{T} \exp\left(\int_0^t ds \{c + dA\right. \\ &\quad \left.+ X_{RR}(s)g[AX_{RR}(s)]A\right) X_{RL}(0). \end{aligned} \quad (\text{D4})$$

The same applies for

$$X_{LR}(t) = X_{LR}(0) \mathcal{T} \exp\left(\int_0^t ds \{d + f[AX_{RR}(s)]\}\right). \quad (\text{D5})$$

Finally, once we have X_{RL} and X_{LR} , Eq. (9) gives X_{LL} as a simple autonomous nonhomogeneous linear system:

$$\begin{aligned} X_{LL}(t) &= e^{ct} X_{LL}(0) \\ &\quad + e^{ct} \int_0^t ds e^{-cs} X_{LR}(s) g[AX_{RR}(s)] A X_{RL}(s). \end{aligned} \quad (\text{D6})$$

Enforcing as an initial condition something proportional to the identity operator simplifies the calculation. $X(0) = \epsilon \in \mathbb{R}$ means $X_{RR}(0) = X_{LL}(0) = \epsilon$ and $X_{RL}(0) = X_{LR}(0) = 0$. Both Eq. (D4) and Eq. (D5) are now identically zero: $X_{RL}(t) = X_{LR}(t) = 0, \forall t$. Equation (D6) reduces to

$$X_{LL}(t) = \epsilon e^{ct}. \quad (\text{D7})$$

Finally, the full solution is $X(t) = X_{RR}(t) + X_{LL}(t)$:

$$X(t) = \xi^{-1}U(t)\xi + \epsilon e^{ct}(1 - \xi^{-1}\xi). \quad (\text{D8})$$

APPENDIX E: SIGNATURES OF PRODUCT MODELS

We prove Eq. (12). The eigenvalues of the product model $A = A_1 \otimes A_2$ are all the possible products of one eigenvalue of the first factor model (A_1), with one eigenvalue of the second factor (A_2). A_1 has rank r_1 and p_1 positive eigenvalues. A_2 has rank r_2 and p_2 positive eigenvalues. A positive eigenvalue of A must be either the product of a positive eigenvalue of A_1 and a positive eigenvalue of A_2 or the product of a negative eigenvalue of A_1 and a negative eigenvalue of A_2 . Hence, $p = p_1 p_2 + (r_1 - p_1)(r_2 - p_2)$. By collecting $r = r_1 r_2$, one gets to Eq. (12).

We prove Eq. (15) for the model in Eq. (13). The negative eigenvalues of the product model are the product of an odd number of negative eigenvalues coming from the

factor models. We define $q_j = r_j - p_j$ as the number of negative eigenvalues of the j th factor and $q = r - p$ as the number of negative eigenvalues of the product model. We can make explicit the multiplicity of eigenvalues by writing

$$r = p + q = \prod_{j=1}^{\tau} r_j = \prod_{j=1}^{\tau} (p_j + q_j). \quad (\text{E1})$$

The expansion of the product would show all the possible ways to pick positive and negative eigenvalues from the factors, and the respective multiplicities. We consider also

$$\prod_{j=1}^{\tau} (p_j - q_j). \quad (\text{E2})$$

In this case the factors of the expanded product are positive if the corresponding multiplicity refers to positive eigenvalues, and they are negative if the corresponding multiplicity refers to negative eigenvalues. Hence,

$$\prod_{j=1}^{\tau} (p_j - q_j) = p - q. \quad (\text{E3})$$

Combining Eqs. (E1) and (E3) we compute p and get to Eq. (15).

Finally, Eq. (22) is simply the special case of Eq. (15) when all the factors have the same rank and signature.

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